

## Quantum tomography, phase-space observables and generalized Markov kernels

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2009 J. Phys. A: Math. Theor. 42 465303

(<http://iopscience.iop.org/1751-8121/42/46/465303>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.156

The article was downloaded on 03/06/2010 at 08:22

Please note that [terms and conditions apply](#).

# Quantum tomography, phase-space observables and generalized Markov kernels

**Juha-Pekka Pellonpää**

Turku Centre for Quantum Physics, Department of Physics and Astronomy, University of Turku, FIN-20014 Turku, Finland

E-mail: [juhpollo@utu.fi](mailto:juhpollo@utu.fi)

Received 11 June 2009, in final form 6 September 2009

Published 22 October 2009

Online at [stacks.iop.org/JPhysA/42/465303](http://stacks.iop.org/JPhysA/42/465303)

## Abstract

We construct a generalized Markov kernel which transforms the observable associated with the homodyne tomography into a covariant phase-space observable with a regular kernel state. Illustrative examples are given in the cases of a ‘Schrödinger cat’ kernel state and the Cahill–Glauber  $s$ -parametrized distributions. Also we consider an example of a kernel state when the generalized Markov kernel cannot be constructed.

PACS numbers: 03.65.–w, 03.67.–a, 42.50.–p

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Quantum homodyne tomography and eight-port homodyne detection are important tools in quantum optics for reconstruction of a quantum state of light (see [16], and references therein). In both measurements, there are associated observables, i.e. normalized positive operator measures (POMs), which give the measurement outcome statistics of the input state. In the eight-port (or double) detection, the corresponding observable is a covariant phase-space observable  $E_K$  generated by a kernel state  $K$  (see, e.g. [11]). Recently, Albini *et al* [2] found a POM  $E_{\text{ht}}$  associated with the quantum homodyne tomography (see also [12]).

The aim of this paper is to find a connection between the two POMs  $E_{\text{ht}}$  and  $E_K$  when  $K$  is regular enough. We will show that  $E_K$  is connected to  $E_{\text{ht}}$  via a generalized Markov kernel. A drawback is that the generalized Markov kernel is not a positive function, so that it cannot be interpreted as a transition probability from  $E_{\text{ht}}$  to  $E_K$ . However,  $E_K$  is a postprocessing of  $E_{\text{ht}}$  and thus subordinate to the tomographic measurement  $E_{\text{ht}}$ .

For a state  $T$ , the generalized Markov kernel associated with  $K$  gives an integral relation between the phase-space probability distribution  $p_K^T$  and the (symplectic) tomogram  $p_{\text{ht}}^T$  of the Wigner function  $W^T$  (see equation (3.3)). Thus, some properties of tomograms can be

transformed to phase-space distributions via generalized Markov kernels; for the theory of tomograms, see e.g. [10].

This work is a direct continuation of our earlier work [15] where we considered only the Husimi  $Q$ -function case  $K = |0\rangle\langle 0|$ . We also showed that there is no hope to represent  $E_{\text{ht}}$  as a postprocessing of  $E_{|0\rangle\langle 0|}$  by using generalized Markov kernels. However, as pointed out in [13], the position and momentum distributions of a vector state  $\psi$  (and possibly all quadrature distributions) can be reconstructed from the probability measure  $Z \mapsto \langle \psi | E_K(Z) \psi \rangle$  when  $K$  and  $\psi$  are regular enough. Hence, there may exist an alternative direct method to reconstruct the statistics of  $E_{\text{ht}}$  from the measurement data of  $E_K$ .

The structure of this paper is as follows. In section 2, we introduce the basic notations, definitions and some well-known results. We also recall the structures of the POMs  $E_{\text{ht}}$  and  $E_K$  associated with the tomographic and eight-port homodyne measurements. We will use the Radon and Hilbert transformations which are probably familiar to readers so that their properties needed in this paper are collected in the appendix.

Section 3 contains the main results of this paper and some illustrative examples. First we note that the tomographic data are somehow ‘sharper’ than the eight-port data for the given input state  $T$ . Then we construct the generalized Markov kernels for POMs  $E_K$  (with suitable kernel states  $K$ ) and then expand them to Hermite and Maclaurin series. Both series are used in the examples. The first example shows the structure of the kernel associated with the phase-space observable generated by a ‘Schrödinger cat’ state. In the second example, kernels are constructed for Cahill–Glauber  $s$ -parametrized distributions (or more precisely, for operator measures associated with them). We will see that although the Husimi  $Q$ -distribution has a very simple kernel, the Wigner function has no kernel at all. Finally, we see that there does not necessarily exist a generalized Markov kernel for an arbitrary kernel state  $K$ .

## 2. Notations and definitions

For any measure space  $(\Omega, \mu)$ , where  $\mu$  is a positive measure on  $\Omega$ , we let  $L^1(\Omega)$  (resp.  $L^2(\Omega)$ ) denote the space of equivalence classes of integrable (resp. square integrable) functions  $\Omega \rightarrow \mathbb{C}$ . In the case of the Hilbert space  $L^2(\Omega)$ , we let  $\langle \cdot | \cdot \rangle_{L^2(\Omega)}$  denote its innerproduct which is assumed to be linear with respect to the second argument. When  $\Omega = \mathbb{R}^n, n = 1, 2$ , the measure  $\mu$  is always the  $n$ -dimensional Lebesgue measure. We let  $\mathcal{S}(\mathbb{R}^n)$  be the Schwartz space of rapidly decreasing smooth complex functions on  $\mathbb{R}^n$ . We identify  $\mathcal{S}(\mathbb{R}^n)$  with a subset of  $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ .

Let  $\mathcal{H}$  be a complex Hilbert space with an innerproduct  $\langle \cdot | \cdot \rangle$  and an orthonormal (number) basis  $\{|n\rangle | n \in \mathbb{N}\}$  where  $\mathbb{N} := \{0, 1, 2, \dots\}$ . Let  $\mathcal{L}(\mathcal{H})$  (resp.  $\mathcal{T}(\mathcal{H})$ ) be the set of bounded operators (resp. trace-class operators) on  $\mathcal{H}$ . The set of states (density matrices), that is, the positive operators  $T \in \mathcal{T}(\mathcal{H})$  of trace 1, is denoted by  $\mathcal{T}(\mathcal{H})_1^+$ . For any  $B \in \mathcal{L}(\mathcal{H})$ , we denote  $B_{mn} := \langle m | B | n \rangle, m, n \in \mathbb{N}$ , so that  $B = \sum_{m,n=0}^{\infty} B_{mn} |m\rangle\langle n|$  where the double series converges with respect to the weak operator topology.

Define the lowering, raising and number operators

$$A := \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle\langle n+1|, \quad A^* := \sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle\langle n|, \quad N := A^* A = \sum_{n=0}^{\infty} n |n\rangle\langle n|,$$

respectively. Physically the Hilbert space  $\mathcal{H}$  and the above operators are associated with a single-mode optical field. We will, *without explicit indication*, use the coordinate

representation, in which  $\mathcal{H}$  is represented as  $L^2(\mathbb{R})$  via the unitary map  $|n\rangle \mapsto h_n$ , where  $h_n \in \mathcal{S}(\mathbb{R})$  is the  $n$ th Hermite function,

$$h_n(x) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{1}{2}x^2} = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{1}{2}x^2} \frac{d^n e^{-x^2}}{dx^n}$$

and  $H_n$  is the  $n$ th Hermite polynomial<sup>1</sup>. Recall that it is customary to denote (formally)  $\psi(x) = \langle x | \psi \rangle$  and  $\int_{\mathbb{R}} |x\rangle \langle x| dx = I$  (the identity operator), that is,  $\langle \psi | \varphi \rangle = \int_{\mathbb{R}} \langle \psi | x \rangle \langle x | \varphi \rangle dx = \int_{\mathbb{R}} \overline{\psi(x)} \varphi(x) dx$  for all  $\psi, \varphi \in \mathcal{H} \cong L^2(\mathbb{R})$ .

Define<sup>2</sup> the position operator  $Q := \frac{1}{\sqrt{2}}(A^* + A)$  and the momentum operator  $P := \frac{i}{\sqrt{2}}(A^* - A)$ , which, in the coordinate representation are the usual multiplication and differentiation operators, respectively:  $(Q\psi)(x) = x\psi(x)$  and  $(P\psi)(x) = -i\frac{d\psi}{dx}(x)$ . For any  $\theta \in [0, 2\pi)$ ,

$$Q_\theta := (\cos \theta)Q + (\sin \theta)P$$

is the (self-adjoint) *rotated quadrature operator* and  $\mathbf{Q}_\theta : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$  is the spectral measure of  $Q_\theta$ ; we denote the Borel  $\sigma$ -algebra of any topological space  $\Omega$  by  $\mathcal{B}(\Omega)$ . It is easy to see that  $\mathbf{Q}_\theta(R) = e^{i\theta N} \mathbf{Q}_0(R) e^{-i\theta N}$  for all  $R \in \mathcal{B}(\mathbb{R})$  and  $\theta \in \mathbb{R}$ .

Define the Weyl operator (or the displacement operator of the complex plane)<sup>3</sup>  $D(z) := e^{\overline{z}A^* - zA}$  for which

$$D(z)^* = D(z)^{-1} = D(-z) \quad \text{and} \quad e^{i\theta N} D(z) e^{-i\theta N} = D(z e^{i\theta}),$$

where  $z \in \mathbb{C}$  and  $\theta \in \mathbb{R}$ . In addition, for all  $z_1, z_2 \in \mathbb{C}$ ,

$$D(z_1 + z_2) = e^{-i \operatorname{Im}(z_1 \overline{z_2})} D(z_1) D(z_2). \tag{2.1}$$

Let  $q, p \in \mathbb{R}$  and  $z = (q + ip)/\sqrt{2}$ . Then

$$D(q, p) := D(z) = e^{ipQ - iqP} = e^{-iqp/2} e^{ipQ} e^{-iqP} = e^{iqp/2} e^{-iqP} e^{ipQ},$$

that is, for all  $\psi \in \mathcal{H} \cong L^2(\mathbb{R})$ ,  $(e^{ipQ}\psi)(x) = e^{ipx}\psi(x)$ ,  $(e^{iqP}\psi)(x) = \psi(x + q)$ , and

$$(D(q, p)\psi)(x) = e^{-iqp/2} e^{ipx}\psi(x - q).$$

Using the polar coordinate parametrization  $z = \rho e^{i\theta}$ ,  $\rho = |z| = \sqrt{\frac{1}{2}(q^2 + p^2)}$ ,  $\theta = \arg z = \arctan(p/q)$ , that is,  $q = \sqrt{2}\rho \cos \theta$ ,  $p = \sqrt{2}\rho \sin \theta$ , one gets

$$e^{i\sqrt{2}\rho Q_\theta} = e^{i(\sqrt{2}\rho \cos \theta)Q + i(\sqrt{2}\rho \sin \theta)P} = e^{iqQ + ipP} = D(-p, q). \tag{2.2}$$

Moreover, the two-dimensional Lebesgue measure is

$$d^2z = \rho d\rho d\theta = \frac{1}{2} d\rho^2 d\theta = \frac{1}{2} dq dp.$$

<sup>1</sup> Hermite polynomials are given by the recursion relation  $H_0(x) = 1$ ,  $H_1(x) = 2x$ , and  $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ , or by the Rodrigue's formula  $H_n(x) = (-1)^n e^{x^2} d^n e^{-x^2}/dx^n$ .

<sup>2</sup>  $\overline{B}$  means the closure of a linear operator  $B$  and  $\overline{z}$  is a complex conjugate of  $z \in \mathbb{C}$ .

<sup>3</sup> Recall that Weyl operators are associated with a unitary representation of the Heisenberg group  $\mathbb{H}$ , or to a projective representation of  $\mathbb{C} \cong \mathbb{R}^2$ .

2.1. Covariant phase-space observables and Wigner functions

Let  $K$  be a trace-class operator on  $\mathcal{H}$ . Define an operator measure  $E_K : \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{L}(\mathcal{H})$  as follows: for all  $Z \in \mathcal{B}(\mathbb{C})$ ,

$$E_K(Z) := \frac{1}{\pi} \int_Z D(z) K D(z)^* d^2z = \frac{1}{2\pi} \int_Z D(q, p) K D(q, p)^* dq dp.$$

Since  $E_K(\mathbb{C}) = (\text{tr } K)I$  we see that  $E_K$  is normalized if and only if  $\text{tr } K = 1$ . Moreover,  $E_K$  is positive if and only if  $K$  is a positive operator. If  $K \in \mathcal{T}(\mathcal{H})_1^+$  we say that the normalized positive operator measure  $E_K$  is a *covariant phase-space observable* generated by the kernel  $K$ . In principle, any covariant phase-space observable can be measured by using double homodyne detection (see, e.g. [11], and references therein).

The parity operator is

$$\Pi := e^{i\pi N} = \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n| = \Pi^+ - \Pi^-$$

where  $\Pi^+ := \sum_{k=0}^{\infty} |2k\rangle\langle 2k|$  and  $\Pi^- := \sum_{k=0}^{\infty} |2k+1\rangle\langle 2k+1|$  are projections onto the closed subspaces  $\mathcal{H}^+ := \Pi^+ \mathcal{H}$  and  $\mathcal{H}^- := \Pi^- \mathcal{H}$  (i.e.,  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ ). Elements of  $\mathcal{H}^+$  (resp.  $\mathcal{H}^-$ ) are even (resp. odd) functions. For any  $\psi \in \mathcal{H} \cong L^2(\mathbb{R})$  we can write

$$(\Pi^\pm \psi)(x) = \frac{1}{2} [\psi(x) \pm \psi(-x)]$$

so that  $(\Pi \psi)(x) = \psi(-x)$  or, formally,  $\Pi = \int_{\mathbb{R}} |x\rangle\langle -x| dx$ .

The Wigner function  $W^T : \mathbb{C} \rightarrow \mathbb{C}$  of a trace-class operator  $T$  is

$$\begin{aligned} W^T(z') &\equiv W^T(q', p') := \frac{1}{\pi} \text{tr} [T D(z') \Pi D(z')^*] = \frac{1}{\pi} \text{tr} [T D(2z') \Pi] \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \text{tr} [T D(-p, q)] e^{-iq'q - ip'p} dq dp \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^\infty \text{tr} [T e^{i\sqrt{2}\rho Q_\theta}] e^{-i\sqrt{2}\rho(q' \cos \theta + p' \sin \theta)} d\rho^2 d\theta, \end{aligned} \quad (2.3)$$

where  $z' = (q' + ip')/\sqrt{2} \in \mathbb{C}$  (see equation (2.2)). Thus, the probability measures  $R \mapsto \text{tr} [T Q_\theta(R)]$ ,  $\theta \in [0, 2\pi)$ , fully determine the Wigner function  $W^T$  of any  $T \in \mathcal{T}(\mathcal{H})_1^+$ . If  $W^T$  is integrable<sup>4</sup> then

$$\int_{\mathbb{R}^2} W^T(q, p) dq dp = 2 \int_{\mathbb{C}} W^T(z) d^2z = \text{tr} [T].$$

Recall that the Wigner function is always square integrable and

$$\text{tr} [TK] = 2\pi \int_{\mathbb{R}^2} W^T(q, p) W^K(q, p) dq dp = 4\pi \int_{\mathbb{C}} W^T(z) W^K(z) d^2z. \quad (2.4)$$

In addition,

$$W^T(q, p) = \frac{1}{\pi} \int_{\mathbb{R}} \tilde{T}(q-x, q+x) e^{2ipx} dx$$

where  $(x, y) \mapsto \tilde{T}(x, y)$  is the integral kernel<sup>5</sup> of a trace-class operator  $T$  considered as an integral operator on  $L^2(\mathbb{R})$ ; usually one denotes formally  $\tilde{T}(x, y) \equiv \langle x|T|y\rangle$ . For future use, we define a linear space

$$\mathcal{T}^S(\mathcal{H}) := \{T \in \mathcal{T}(\mathcal{H}) | \tilde{T} \in \mathcal{S}(\mathbb{R}^2)\}.$$

<sup>4</sup> This need not hold even for all pure states  $T = |\eta\rangle\langle\eta|$  (see example 3).

<sup>5</sup> That is, for all  $\psi \in L^2(\mathbb{R})$ ,  $(T\psi)(x) = \int_{\mathbb{R}} \tilde{T}(x, y)\psi(y)dy$  for dx-almost all  $x \in \mathbb{R}$ . More precisely,  $\tilde{T}$  is an equivalence class of functions which satisfy this condition.

Note that the mapping  $\mathcal{T}^S(\mathcal{H}) \ni T \mapsto W^T \in \mathcal{S}(\mathbb{R}^2)$  is bijective (see, e.g. [2, proposition 4]).

For any  $K$  and  $T \in \mathcal{T}(\mathcal{H})$  we get

$$\text{tr}[TE_K(Z)] = \frac{1}{\pi} \int_Z \text{tr}[TD(z_1)KD(z_1)^*] d^2z_1 = \frac{1}{\pi} \int_Z p_K^T(z_1) d^2z_1$$

where, by equations (2.4) and (2.1), the density

$$\begin{aligned} p_K^T(z_1) &:= \text{tr}[TD(z_1)KD(z_1)^*] = 4\pi \int_{\mathbb{C}} W^{TD(z_1)}(z_2)W^{KD(z_1)^*}(z_2) d^2z_2 \\ &= 4\pi \int_{\mathbb{C}} W^T(z_1/2+z_2)W^K(-z_1/2+z_2) d^2z_2 = 4\pi \int_{\mathbb{C}} W^T(z)W^K(z-z_1) d^2z \end{aligned}$$

and  $p_K^T(z_1) = p_K^K(-z_1)$ . Note that  $p_K^T$  is a well-defined continuous function even if  $K$  is only a bounded operator. Thus, from equation (2.3), one has

$$W^T = \frac{1}{2\pi} p_{2\Pi}^T.$$

### 2.2. The POM associated with homodyne tomography

Let  $\Omega := [0, \pi) \times \mathbb{R}$  so that  $\Omega$  gives a parametrization for the projective space  $\mathbb{P}^2$  (see the appendix). Define the normalized positive operator measure  $\tilde{E}_{\text{ht}} : \mathcal{B}(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$  associated with homodyne tomography [2] as

$$\begin{aligned} \tilde{E}_{\text{ht}}(\Theta \times R) &:= \frac{1}{\pi} \int_{\Theta} Q_{\theta}(R) d\theta \\ &= \frac{1}{\pi} \int_{\Theta} e^{i\theta N} Q_0(R) e^{-i\theta N} d\theta, \quad \Theta \in \mathcal{B}([0, \pi)), \quad R \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

Obviously, one can define similarly a POM associated with the double covering  $\mathbb{S}^1 \times \mathbb{R}$  of  $\mathbb{P}^2$  as

$$\begin{aligned} E_{\text{ht}}(\Theta \times R) &:= \frac{1}{2\pi} \int_{\Theta} Q_{\theta}(R) d\theta \\ &= \frac{1}{2} \tilde{E}_{\text{ht}}(\Theta \cap [0, \pi) \times R) + \frac{1}{2} \tilde{E}_{\text{ht}}([\Theta \cap [\pi, 2\pi)) - \pi] \times (-R)), \end{aligned}$$

where now  $\Theta \in \mathcal{B}([0, 2\pi))$  and  $R \in \mathcal{B}(\mathbb{R})$ .

For any  $T \in \mathcal{T}(\mathcal{H})$ , let  $p_{\text{ht}}^T(\theta, r)$  be the density of the complex measure  $\text{tr}[TE_{\text{ht}}]$  with respect to  $(2\pi)^{-1} d\theta dr$  [2]. For example, since

$$\langle n | E_{\text{ht}}(\Theta \times R) | m \rangle = \frac{1}{2\pi} \int_{\Theta} e^{i(n-m)\theta} d\theta \int_R h_n(r)h_m(r) dr,$$

it follows that

$$p_{\text{ht}}^{|m\rangle\langle n|}(\theta, r) = e^{i(n-m)\theta} h_n(r)h_m(r) = e^{i(n-m)\theta} p_{\text{ht}}^{|m\rangle\langle n|}(0, r).$$

The next proposition shows that if  $T$  is regular enough, one can express  $p_{\text{ht}}^T$  using the matrix elements of  $T$ .

**Proposition 1.** For any  $T \in \mathcal{T}^S(\mathcal{H})$  we have

(1) the double sequence  $(T_{mn})_{m,n \in \mathbb{N}}$  is rapidly decreasing,<sup>6</sup>

<sup>6</sup> That is, the series  $\sum_{m,n=0}^{\infty} m^{2j} n^{2k} |T_{mn}|^2$  converge for all  $j, k \in \mathbb{N}$ .

- (2) the integral kernel  $\tilde{T}(x, y) = \sum_{m,n=0}^{\infty} T_{mn} h_m(x) h_n(y)$  where the double series converges absolutely for all  $x, y \in \mathbb{R}$ ;<sup>7</sup>
- (3)  $p_{\text{ht}}^T(0, r) = \sum_{m,n=0}^{\infty} T_{mn} h_n(r) h_m(r) = \tilde{T}(r, r)$  for (almost) all  $r \in \mathbb{R}$ ;
- (4) the function  $r \mapsto \tilde{T}(r, r)$  belongs to  $\mathcal{S}(\mathbb{R})$ ;
- (5)  $e^{-i\theta N} T e^{i\theta N} \in \mathcal{T}^{\mathcal{S}}(\mathcal{H})$  for all  $\theta \in \mathbb{R}$ ;
- (6)  $p_{\text{ht}}^T(\theta, r) = p_{\text{ht}}^{e^{-i\theta N} T e^{i\theta N}}(0, r) = \sum_{m,n=0}^{\infty} T_{mn} e^{i(n-m)\theta} h_n(r) h_m(r)$  for (almost) all  $\theta \in [0, 2\pi)$  and  $r \in \mathbb{R}$ ;
- (7) the Wigner function  $W^T \in \mathcal{S}(\mathbb{R}^2)$ ;
- (8) the Radon transform  $(\text{RW}^T)(\theta, r) = p_{\text{ht}}^T(\theta, r)$  for (almost) all  $\theta \in [0, 2\pi)$  and  $r \in \mathbb{R}$ .

**Proof.** (1) is a standard result, see e.g. [19, theorem V.13]. Item (2) follows from the facts that

$$\sup\{|h_n(x)| \geq 0 | n \in \mathbb{N}, x \in \mathbb{R}\} < 1$$

(see, equation (22.14.17) of [1] in p 787) and  $\sum_{m,n=0}^{\infty} |T_{mn}| < \infty$ ; moreover, for all  $\psi = \sum_{n=0}^{\infty} c_n h_n$ ,  $\sum_{n=0}^{\infty} |c_n|^2 < \infty$ ,

$$\int_{\mathbb{R}} \tilde{T}(x, y) \psi(y) dy = \sum_{m,n=0}^{\infty} T_{mn} h_m(x) \int_{\mathbb{R}} h_n(y) \psi(y) dy = \sum_{m=0}^{\infty} h_m(x) \sum_{n=0}^{\infty} T_{mn} c_n = (T\psi)(x)$$

for almost all  $x \in \mathbb{R}$ . Item (3) follows from

$$\text{tr}[TQ_0(R)] = \int_R \tilde{T}(r, r) dr, \quad R \in \mathcal{B}(\mathbb{R}).$$

Since, for any  $f \in \mathcal{S}(\mathbb{R}^2)$ , the mapping  $r \mapsto f(r, r)$  belongs to  $\mathcal{S}(\mathbb{R})$ , item (4) follows. Item (5) is obvious and (6) is easy to see since  $Q_{\theta}(R) = e^{i\theta N} Q_0(R) e^{-i\theta N}$  for all  $R \in \mathcal{B}(\mathbb{R})$ ,  $\theta \in \mathbb{R}$ . Item (7) is a standard result (see, e.g. [2, proposition 4]) and (8) is proved in [2, proposition 2]. □

### 3. Results

In this section, we will find different connections between phase-space observables  $E_K$ ,  $K \in \mathcal{T}(\mathcal{H})_1^+$  and  $E_{\text{ht}}$ , the observable associated with homodyne tomography. Our results are based on the properties of the Radon transform  $\mathbb{R}$  and on the Hilbert transform  $\mathbb{H}$  which are shortly introduced in the appendix. The main mathematical results to be used are the *convolution theorem* (A.1), *Radon inversion theorem* (A.3) and *Plancherel formula* (A.4).

#### 3.1. Convolution theorem and Radon inversion theorem

Let  $T, K \in \mathcal{T}(\mathcal{H})$ . The density of  $Z \mapsto \text{tr}[TE_K(Z)]$  at  $z_1 = (q_1 + ip_1)/\sqrt{2} \in \mathbb{C}$  is

$$\begin{aligned} p_K^T(z_1) &:= \text{tr}[TD(z_1)KD(z_1)^*] = 4\pi \int_{\mathbb{C}} W^T(z)W^K(z - z_1) d^2z \\ &= 2\pi \iint_{\mathbb{R}^2} W^T(q, p)W^K(q - q_1, p - p_1) dq dp \\ &= 2\pi \iint_{\mathbb{R}^2} \tilde{W}^K(q_1 - q, p_1 - p)W^T(q, p) dq dp \\ &= 2\pi(W^T * \tilde{W}^K)(q_1, p_1) = 2\pi(\tilde{W}^K * W^T)(q_1, p_1) \end{aligned}$$

<sup>7</sup> Obviously, here we have chosen a representative of the equivalence class of kernels  $\tilde{T}$ ; if one chooses another representative then the results of this proposition hold only for *almost* all points.

where  $\tilde{W}^K(q, p) := W^K(-q, -p)$ . If  $W^T$  and  $W^K$  are integrable then it follows from [2, proposition 2] that

$$(\mathbb{R}W^T)(\theta, r) = p_{\text{ht}}^T(\theta, r)$$

for  $d\theta dr$ -almost all  $(\theta, r)$ , and similarly for  $W^K$ . Moreover, one gets from the convolution theorem (A.1) that

$$(\mathbb{R}p_K^T)(\theta, r) = 2\pi \int_{\mathbb{R}} p_{\text{ht}}^T(\theta, s) p_{\text{ht}}^K(\theta, s - r) ds \tag{3.1}$$

for  $d\theta dr$ -almost all  $(\theta, r)$ . Note that, *formally*, replacing  $p_K^T$  with  $W^T = \frac{1}{2\pi} p_{2\pi}^T$  (which is only a quasiprobability distribution for non-Gaussian states) in the above equation, we get

$$(\mathbb{R}W^T)(\theta, r) = p_{\text{ht}}^T(\theta, r) = \int_{\mathbb{R}} p_{\text{ht}}^T(\theta, s) \delta(s - r) ds$$

and  $p_{\text{ht}}^{2\pi}(\theta, r) = \delta(r)$  where  $\delta(s - r)ds$  is the Dirac measure concentrated on  $r$ , so that, the phase-space  $p_K^T$  distribution is a kind of a ‘smoothed version’ of quadrature data (associated with  $W^T$ ).

Keeping equation (3.1) in mind, one gets from Radon inversion theorem (A.3) that

$$p_K^T = \frac{1}{2\pi} \mathbb{R}^* \Lambda \underbrace{(\mathbb{R}p_K^T)}_{p_{\text{ht}}^T, p_{\text{ht}}^K} \tag{3.2}$$

if  $p_K^T$  is a Schwartz function<sup>8</sup>. Thus, the probability distribution  $p_K^T$  where  $T, K \in \mathcal{T}^S(\mathcal{H}) \cap \mathcal{T}(\mathcal{H})_1^+$  can be constructed by using homodyne tomography data for both  $T$  and  $K$ . Physically, this means that to *reconstruct* the measurement outcome statistics  $p_K^T$  of an eight-port homodyne detector, one needs to measure the statistics  $p_{\text{ht}}^T$  and  $p_{\text{ht}}^K$  separately by using balanced homodyne detection for quadratures, and then use formulae (3.1) and (3.2). In some cases, this could be easier in the sense that if one needs to prepare or measure the kernel state  $K$ , one can use quadrature measurements (tomography) to get  $p_{\text{ht}}^K$  (and find the state  $K$  if it is unknown *a priori*). Then  $p_K^T$  can be obtained directly either by using eight-port detection (since  $K$  is known) or by measuring  $p_{\text{ht}}^T$  via quantum tomography. Since that  $p_K^T$  is (almost) symmetric in the interchange of  $K$  and  $T$ , that is,  $p_K^T(z_1) = p_T^K(-z_1)$ , *mathematically*, the signal state  $T$  and the kernel state  $K$  are in the same role in eight-port homodyne detection.

### 3.2. Plancherel formula and Markov kernels

Let  $K \in \mathcal{T}(\mathcal{H})$  be such that  $W^K \in L^1(\mathbb{R}^2)$  and  $(q, p) \in \mathbb{R}^2$ ,  $(q_1, p_1) \in \mathbb{R}^2$  and  $\theta \in [0, 2\pi)$ . If one denotes  $W_{q_1, p_1}^K(q, p) := W^K(q - q_1, p - p_1)$  and defines parameters  $a := q_1 \cos \theta + p_1 \sin \theta$ ,  $b := -q_1 \sin \theta + p_1 \cos \theta$ , one gets

$$\begin{aligned} (\mathbb{R}W_{q_1, p_1}^K)(\theta, r) &= \int_{\mathbb{R}} W_{q_1, p_1}^K(r \cos \theta - t \sin \theta, r \sin \theta + t \cos \theta) dt \\ &= \int_{\mathbb{R}} W^K(r \cos \theta - t \sin \theta - q_1, r \sin \theta + t \cos \theta - q_2) dt \\ &= \int_{\mathbb{R}} W^K((r - a) \cos \theta - (t - b) \sin \theta, (r - a) \sin \theta + (t - b) \cos \theta) d(t - b) \\ &= (\mathbb{R}W^K)(\theta, r - a) = (\mathbb{R}W^K)(\theta, r - q_1 \cos \theta - p_1 \sin \theta) \\ &= p_{\text{ht}}^K(\theta, r - q_1 \cos \theta - p_1 \sin \theta). \end{aligned}$$

<sup>8</sup> This holds, e.g., when  $T, K \in \mathcal{T}^S(\mathcal{H})$  since then  $p_K^T$  is a convolution of Schwartz functions  $W^T$  and  $\tilde{W}^K$  and thus a Schwartz function (see, proposition 1).



Assume then that  $K, T \in \mathcal{T}^S(\mathcal{H})$  and define  $M_{q_1, p_1}^K := \Lambda(\mathbf{R}W_{q_1, p_1}^K)$  (see equation (A.2)). Since  $\mathbf{R}$  maps  $\mathcal{S}(\mathbb{R}^2)$  to  $\mathcal{S}_H(\mathbb{P}^2)$ ,  $(\Lambda\varphi)(\theta, r) = \pi \frac{d}{dr}(\mathbf{H}\varphi_\theta)(r)$ , and the Hilbert transform  $\mathbf{H}$  commutes with derivation, one sees that  $M_{q_1, p_1}^K(\theta, r)$  is smooth with respect to all variables  $q_1, q_2, \theta$ , and  $r$  (see the appendix). Since

$$p_K^T(q_1, p_1) = 2\pi \iint_{\mathbb{R}^2} W^T(q, p)W^K(q - q_1, p - p_1) dq dp$$

we get from Plancherel formula (A.4) that

$$p_K^T(q_1, p_1) = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} p_{\text{ht}}^T(\theta, r)M_{q_1, p_1}^K(\theta, r) d\theta dr. \tag{3.3}$$

Hence, for any  $Z \in \mathcal{B}(\mathbb{R}^2)$ ,

$$\begin{aligned} \text{tr}[TE_K(Z)] &= \frac{1}{2\pi} \int_Z p_K^T(q_1, p_1) dq_1 dp_1 \\ &= \frac{1}{2\pi} \int_Z \int_0^{2\pi} \int_{\mathbb{R}} M_{q_1, p_1}^K(\theta, r) d\text{tr}[TE_{\text{ht}}(\theta, r)] dq_1 dp_1. \end{aligned}$$

If  $Z$  is compact, it follows from Fubini's theorem that one can change the order of the above integrations and get for operator measures  $E_K$  and  $E_{\text{ht}}$  (by the density of  $\mathcal{T}^S(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ ) that

$$E_K(Z) = \int_0^{2\pi} \int_{\mathbb{R}} \underbrace{\left[ \frac{1}{2\pi} \int_Z M_{q_1, p_1}^K(\theta, r) dq_1 dp_1 \right]}_{=: \mathcal{M}_K(Z; \theta, r)} dE_{\text{ht}}(\theta, r) = \int_0^{2\pi} \int_{\mathbb{R}} \mathcal{M}_K(Z; \theta, r) dE_{\text{ht}}(\theta, r)$$

(weakly) for all compact  $Z \subset \mathbb{R}^2$ , where  $\mathcal{M}_K$  is a *generalized Markov kernel*. Note that  $\mathcal{M}_K$  is *not* a Markov kernel<sup>9</sup> for the following reasons: (a)  $\mathcal{M}_K(Z; \theta, r)$  is not defined for *all* Borel<sup>10</sup> sets  $Z \in \mathcal{B}(\mathbb{R}^2)$ , (b)  $\mathcal{M}_K$  is not necessarily a nonnegative function (see examples 1 and 2). However, for compact sets  $Z$ , the map  $(\theta, r) \mapsto \mathcal{M}_K(Z; \theta, r)$  is measurable (even continuous since  $M_{q_1, p_1}^K(\theta, r)$  is smooth).

Due to the lack of positivity,  $\mathcal{M}_K(Z; \theta, r)$  is not a true conditional or transition probability. Still one could say that the measurement  $E_K$  is subordinate to the measurement  $E_{\text{ht}}$  (see, [9]) or  $E_K$  is a postprocessing of  $E_{\text{ht}}$ .

Since  $p_K^T(q_1, p_1) = \sum_{m, n=0}^{\infty} K_{mn} p_{|m\rangle\langle n|}^T(q_1, p_1)$  for all  $T \in \mathcal{T}(\mathcal{H})$  (where the double series converges absolutely) it follows that (weakly)

$$\begin{aligned} \int_0^{2\pi} \int_{\mathbb{R}} \mathcal{M}_K(Z; \theta, r) dE_{\text{ht}}(\theta, r) &= E_K(Z) = \sum_{m, n=0}^{\infty} K_{mn} E_{|m\rangle\langle n|}(Z) \\ &= \sum_{m, n=0}^{\infty} K_{mn} \int_0^{2\pi} \int_{\mathbb{R}} \mathcal{M}_{|m\rangle\langle n|}(Z; \theta, r) dE_{\text{ht}}(\theta, r) \end{aligned}$$

for all compact  $Z \subset \mathbb{R}^2$ . Thus, one needs to find the density  $M_{q_1, p_1}^K(\theta, r)$  of the Markov kernel only in the case of  $K = |m\rangle\langle n|$  where  $m, n \in \mathbb{N}$ . Before doing that we study some properties of the densities which are immediate from the definitions.

We note that when  $q_1 = 0 = p_1$ , one gets  $p_K^T(0, 0) = \text{tr}[TK]$  where  $T, K \in \mathcal{T}(\mathcal{H})$ . Thus, if  $T, K \in \mathcal{T}^S(\mathcal{H})$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} M_{0,0}^K(\theta, r) p_{\text{ht}}^T(\theta, r) d\theta dr = \text{tr}[TK].$$

<sup>9</sup> Recall that if  $\mathcal{M}_K$  was a Markov kernel, then  $\mathcal{M}_K$  should satisfy the following conditions: (i)  $\forall (\theta, r)$ , the map  $\mathcal{B}(\mathbb{R}^2) \ni Z \mapsto \mathcal{M}_K(Z; \theta, r) \in [0, 1]$  is a probability measure, (ii)  $\forall Z \in \mathcal{B}(\mathbb{R}^2)$ , the map  $(\theta, r) \mapsto \mathcal{M}_K(Z; \theta, r)$  is measurable.

<sup>10</sup> Note that the effects  $E_K(Z)$  of compact sets  $Z$  fully determine the POM  $E_K, K \in \mathcal{T}^S(\mathcal{H}) \cap \mathcal{T}(\mathcal{H})_1^\dagger$ .

Especially, by choosing  $K = |m\rangle\langle n|$ ,  $m, n \in \mathbb{N}$ , one recovers the matrix element  $T_{nm}$  from the tomographic data  $p_{\text{ht}}^T$  as follows:

$$T_{nm} = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} M_{0,0}^{|m\rangle\langle n|}(\theta, r) p_{\text{ht}}^T(\theta, r) \, d\theta \, dr$$

(see [16, section 5], and references therein). If  $T = |k\rangle\langle l|$ ,  $k, l \in \mathbb{N}$ , then

$$\begin{aligned} K_{lk} &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} M_{0,0}^K(\theta, r) p_{\text{ht}}^{[k]\langle l]}(\theta, r) \, d\theta \, dr \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} M_{0,0}^K(\theta, r) e^{i(l-k)\theta} h_l(r) h_k(r) \, d\theta \, dr \end{aligned}$$

since  $p_{\text{ht}}^{[k]\langle l]}(\theta, r) = e^{i(l-k)\theta} h_l(r) h_k(r)$ . Combining the above two results, we get the *orthogonality condition*:

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} M_{0,0}^{|m\rangle\langle n|}(\theta, r) e^{i(l-k)\theta} h_l(r) h_k(r) \, d\theta \, dr = \delta_{m,l} \delta_{n,k} \tag{3.4}$$

(where  $\delta_{m,l}$  is the Kronecker delta). Next we calculate the structure of  $M_{q_1, p_1}^K(\theta, r)$  in the case of  $K = |m\rangle\langle n|$ .

Recall that  $M_{q_1, p_1}^K = \Lambda(\mathbf{R}W_{q_1, p_1}^K)$  where  $(\mathbf{R}W_{q_1, p_1}^K)(\theta, r) = p_{\text{ht}}^K(\theta, r - a)$  and  $a = q_1 \cos \theta + p_1 \sin \theta$ . Since  $p_{\text{ht}}^{|m\rangle\langle n|}(\theta, r) = e^{i(n-m)\theta} h_n(r) h_m(r)$  one gets

$$(\mathbf{R}W_{q_1, p_1}^{|m\rangle\langle n|})(\theta, r) = e^{i(n-m)\theta} h_n(r - a) h_m(r - a).$$

Since the Hilbert transform  $H$  is shift invariant and commutes with derivation (see the appendix) one needs only to find  $H(h_n h_m)$ .

Consider first the case  $n = m = 0$  (i.e. the Husimi  $Q$ -function case where  $K = |0\rangle\langle 0|$ ). Then

$$h_0(x) = \frac{1}{\sqrt[3]{\pi}} e^{-\frac{1}{2}x^2} \quad \text{and} \quad h_0(x)^2 = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

Let  $\text{daw}(r) := e^{-r^2} \int_0^r e^{x^2} \, dx$  be the *Dawson's integral* which is related to the imaginary error function  $\text{erfi}$  in the obvious way (see, e.g. [1, 21]). The result of the next lemma is well known but, for completeness, we give an alternative proof for it.

**Lemma 1.** For all  $r \in \mathbb{R}$ ,

$$[H(h_0^2)](r) = \frac{2}{\pi} e^{-r^2} \int_0^r e^{x^2} \, dx = \frac{2}{\pi} \text{daw}(r) = \frac{1}{\sqrt{\pi}} e^{-r^2} \text{erfi}(r).$$

**Proof.** Let

$$f(r) := [H(h_0^2)](r) = \frac{1}{\pi^{3/2}} \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{e^{-x^2}}{r-x} \, dx.$$

Since

$$\begin{aligned} \frac{df(r)}{dr} &= \left[ H \left( \frac{dh_0^2}{dx} \right) \right](r) = \frac{1}{\pi^{3/2}} \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{-2xe^{-x^2}}{r-x} \, dx \\ &= \frac{2}{\pi^{3/2}} \left[ \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{(r-x)e^{-x^2}}{r-x} \, dx - \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{re^{-x^2}}{r-x} \, dx \right] \\ &= \frac{2}{\pi^{3/2}} \left[ \int_{\mathbb{R}} e^{-x^2} \, dx - r \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{e^{-x^2}}{r-x} \, dx \right] = \frac{2}{\pi} - 2rf(r) \end{aligned}$$

and

$$\frac{d}{dr} \frac{2}{\pi} \text{daw}(r) = \frac{2}{\pi} - 2r \frac{2}{\pi} \text{daw}(r)$$

it follows that

$$f(r) = \frac{2}{\pi} \text{daw}(r) + c e^{-r^2}$$

where  $c \in \mathbb{R}$ . But

$$f(0) = \frac{1}{\pi^{3/2}} \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |x| > \epsilon}} \frac{e^{-x^2}}{-x} dx = \frac{1}{\pi^{3/2}} \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{\infty} \frac{e^{-(-y)^2} - e^{-y^2}}{y} dy = 0 = 0 + c$$

so that  $c = 0$  and the lemma follows. □

Since  $(\mathbb{R}W_{q_1, p_1}^{(0)|0\rangle})(\theta, r) = [h_0(r - a)]^2$ ,  $(\Delta\varphi)(\theta, r) = \pi \frac{d}{dr}(\mathbb{H}\varphi_\theta)(r)$ , and  $M_{q_1, p_1}^{(0)|0\rangle} = \Lambda(\mathbb{R}W_{q_1, p_1}^{(0)|0\rangle})$ , the above lemma implies that

$$\begin{aligned} M_{q_1, p_1}^{(0)|0\rangle}(\theta, r) &= 2 \frac{\partial}{\partial r} \text{daw}(r - q_1 \cos \theta - p_1 \sin \theta) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^k (2k)!} H_{2k}(r - q_1 \cos \theta - p_1 \sin \theta) \end{aligned}$$

is an analytic function vanishing at infinity  $r \rightarrow \pm\infty$  (see, lemma 1 of [12]). Moreover,

$$E_{|0\rangle\langle 0|}(Z) = \int_0^{2\pi} \int_{\mathbb{R}} \left[ \frac{1}{2\pi} \int_Z M_{q_1, p_1}^{(0)|0\rangle}(\theta, r) dq_1 dp_1 \right] dE_{\text{ht}}(\theta, r)$$

(see a direct verification from [15]). Next we consider the general case when  $K = |m\rangle\langle n|$  where  $m, n \in \mathbb{N}$ .

**Proposition 2.** For all  $r \in \mathbb{R}$ ,

$$[\mathbb{H}(h_n h_m)](r) = \frac{(-1)^{n+m}}{\pi} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} \frac{d^{n+m-2v}(2\text{daw}(r))}{dr^{n+m-2v}}$$

**Proof.** By the *Feldheim's identity* [7, equation (1.4)]

$$H_n(r)H_m(r) = n!m! \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} H_{n+m-2v}(r)$$

and *Rodrigue's formula*

$$H_k(r) = (-1)^k e^{r^2} \frac{d^k e^{-r^2}}{dr^k}$$

one can write

$$\begin{aligned} h_n(r)h_m(r) &= \frac{1}{\sqrt{2^{n+m}n!m!\pi}} H_n(r)H_m(r) e^{-r^2} \\ &= \frac{n!m!}{\sqrt{2^{n+m}n!m!\pi}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} H_{n+m-2v}(r) e^{-r^2} \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} (-1)^{n+m-2v} \frac{d^{n+m-2v} e^{-r^2}}{dr^{n+m-2v}}. \end{aligned}$$

Since  $e^{-r^2} = \sqrt{\pi} h_0(r)^2$  and  $H$  commutes with derivation, one gets

$$H(h_n h_m) = (-1)^{n+m} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} \frac{d^{n+m-2v} H(h_0^2)}{dr^{n+m-2v}}$$

and the proposition follows from lemma 1. □

For any smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$  we let  $f^{(s)}$  denote the  $s$ th derivative function of  $f$  (and  $f^{(0)} := f$ ). Let  $Y := 2 \operatorname{daw}^{(1)}$ . As shown in the appendix of [12], for all  $p \in \mathbb{N}$ ,

$$Y^{(2p)}(r) = (-1)^p 2^p \sum_{k=0}^{\infty} \frac{(-1)^k (k+p)!}{2^k (2k)!} H_{2k}(r), \tag{3.5}$$

$$Y^{(2p+1)}(r) = (-1)^{p+1} 2^p \sum_{k=0}^{\infty} \frac{(-1)^k (k+p+1)!}{2^k (2k+1)!} H_{2k+1}(r) \tag{3.6}$$

for all  $r \in \mathbb{R}$ . Finally, we are ready to calculate the density  $M_{q_1, p_1}^{(m)\langle n|}$  of the kernel associated with  $|m\rangle\langle n|$ : from the previous proposition one gets

$$\begin{aligned} M_{q_1, p_1}^{(m)\langle n|}(\theta, r) &= \pi e^{i(n-m)\theta} \frac{\partial [H(h_n h_m)](r-a)}{\partial r} \\ &= e^{i(n-m)\theta} (-1)^{n+m} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} Y^{(n+m-2v)}(r-a) \end{aligned} \tag{3.7}$$

where  $a = q_1 \cos \theta + p_1 \sin \theta$ . Immediately one sees that  $\mathbb{R} \ni r \mapsto M_{0,0}^{(m)\langle n|}(0, r) \in \mathbb{R}$  is analytic and vanishes at infinity  $r \rightarrow \pm\infty$ . Moreover,

$$M_{q_1, p_1}^{(m)\langle n|}(\theta, r) = e^{i(n-m)\theta} M_{0,0}^{(m)\langle n|}(0, r - q_1 \cos \theta - p_1 \sin \theta) = \overline{M_{q_1, p_1}^{(n)\langle m|}(\theta, r)}$$

and the orthogonality relation (3.4) reduces to

$$\int_{\mathbb{R}} M_{0,0}^{(m)\langle n|}(0, r) h_l(r) h_k(r) dr \underbrace{\frac{1}{2\pi} \int_0^{2\pi} e^{i(l-m-k+n)\theta} d\theta}_{=\delta_{l-m-k+n,0}} = \delta_{m,l} \delta_{n,k}.$$

Obviously, without restricting generality, we may assume that  $\theta = 0, q_1 = 0$  and  $p_1 = 0$  in the next theorem.

**Theorem 1.** *If  $n+m$  is even, then*

$$M_{0,0}^{(m)\langle n|}(0, r) = \frac{(-1)^{(n+m)/2}}{\sqrt{n!m!}} \sum_{k=(n+m)/2}^{\infty} \frac{(-1)^k (k + \frac{1}{2}(n-m))! (k + \frac{1}{2}(m-n))!}{2^k (2k)! (k - \frac{1}{2}(n+m))!} H_{2k}(r),$$

*and if  $n+m$  is odd, then*

$$\begin{aligned} M_{0,0}^{(m)\langle n|}(0, r) &= \frac{(-1)^{(n+m-1)/2}}{\sqrt{2n!m!}} \\ &\times \sum_{k=(n+m-1)/2}^{\infty} \frac{(-1)^k (k + \frac{1}{2}(n-m+1))! (k + \frac{1}{2}(m-n+1))!}{2^k (2k+1)! (k - \frac{1}{2}(n+m-1))!} H_{2k+1}(r) \end{aligned}$$

for all  $r \in \mathbb{R}$ .

**Proof.** Suppose that  $n + m$  is even so that one can write  $n + m = 2l$  where  $l \in \mathbb{N}$ . Then from equations (3.5) and (3.7) one gets

$$\begin{aligned} M_{0,0}^{(m)(n)}(0, r) &= (-1)^{n+m} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} Y^{(n+m-2v)}(r) \\ &= \frac{\sqrt{n!m!}}{2^l} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} (-1)^{l-v} 2^{l-v} \sum_{k=0}^{\infty} \frac{(-1)^k (k+l-v)!}{2^k (2k)!} H_{2k}(r) \\ &= \sqrt{n!m!} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k (2k)!} H_{2k}(r) \sum_{v=0}^{\min\{n,m\}} \frac{(-1)^{l-v} (k+l-v)!}{v!(n-v)!(m-v)!} \\ &= \sqrt{n!m!} (-1)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k (2k)!} H_{2k}(r) \frac{(k+l-m)!}{n!} \\ &\quad \times \underbrace{\sum_{v=0}^{\min\{n,m\}} (-1)^v \binom{n}{v} \binom{k+l-v}{m-v}}_{= \binom{k+l-n}{m}} \text{ (see [20, equation (5)] in p 8)} \\ &= \sqrt{n!m!} (-1)^l \sum_{k=l}^{\infty} \frac{(-1)^k}{2^k (2k)!} H_{2k}(r) \frac{(k+l-m)!(k+l-n)!}{n!m!(k+l-n-m)!} \\ &= \frac{(-1)^l}{\sqrt{n!m!}} \sum_{k=l}^{\infty} \frac{(-1)^k}{2^k (2k)!} \frac{(k+l-m)!(k+l-n)!}{(k+l-n-m)!} H_{2k}(r) \end{aligned}$$

and the first equation follows by substituting  $l = (n + m)/2$  into the above equation.

Similarly, if  $n + m = 2l + 1$ ,  $l \in \mathbb{N}$ , then from (3.6) and (3.7) it follows that

$$\begin{aligned} M_{0,0}^{(m)(n)}(0, r) &= (-1)^{n+m} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} Y^{(n+m-2v)}(r) \\ &= \sqrt{\frac{n!m!}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k (2k+1)!} H_{2k+1}(r) \sum_{v=0}^{\min\{n,m\}} \frac{(-1)^{l-v} (k+l-v+1)!}{v!(n-v)!(m-v)!} \\ &= \sqrt{\frac{n!m!}{2}} (-1)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k (2k+1)!} H_{2k+1}(r) \frac{(k+1+l-m)!(k+1+l-n)!}{n!} \binom{k+1+l-n}{m} \\ &= \frac{(-1)^l}{\sqrt{2n!m!}} \sum_{k=l}^{\infty} \frac{(-1)^k}{2^k (2k+1)!} \frac{(k+1+l-m)!(k+1+l-n)!}{(k+1+l-n-m)!} H_{2k+1}(r) \end{aligned}$$

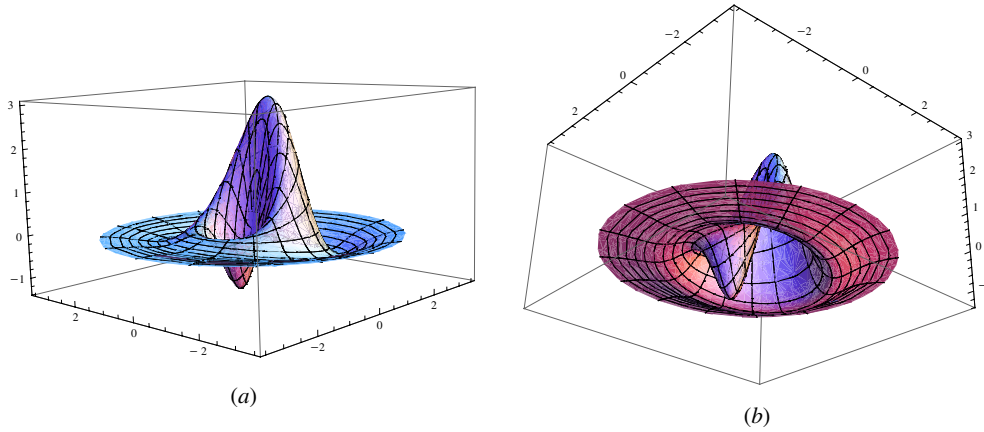
and the second equation has been proved. □

**Remark 1.** Since  $Y$  is an analytic function (see, equation (A.6) of [12]) one sees from equation (3.7) that  $r \mapsto M_{0,0}^{(m)(n)}(0, r)$  is an analytic function. Indeed, its Maclaurin series is easy to compute by using the series

$$Y(r) = 2 \sum_{u=0}^{\infty} \frac{(-1)^u u!}{(2u)!} (2r)^{2u} \tag{3.8}$$

which implies that, for all  $p \in \mathbb{N}$ ,

$$Y^{(p)}(r) = 2 \sum_{\substack{u \in \mathbb{N} \\ u \geq p/2}} \frac{(-1)^u u!}{(2u-p)!} 2^{2u} r^{2u-p}.$$



**Figure 1.** The density  $M_{0,0}^{|\psi\rangle\langle\psi|}(\theta, r)$  of the generalized Markov kernel in the case of a Schrödinger cat state  $\psi = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ .

For example, if  $n + m = 2l, l \in \mathbb{N}$ , one gets

$$\begin{aligned} M_{0,0}^{(m)\langle n|}(0, r) &= (-1)^{n+m} \sqrt{\frac{n!m!}{2^{n+m}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} Y^{(n+m-2v)}(r) \\ &= \sqrt{\frac{n!m!}{2^{2l}}} \sum_{v=0}^{\min\{n,m\}} \frac{2^v}{v!(n-v)!(m-v)!} 2 \sum_{u=l-v}^{\infty} \frac{(-1)^u u!}{(2u-2l+2v)!} 2^{2u} r^{2(u-l+v)} \\ &= 2\sqrt{n!m!} \sum_{t=0}^{\infty} \frac{(-1)^t (2r)^{2t}}{(2t)!} \sum_{v=0}^{\min\{n,m\}} \frac{(-2)^{l-v} (l-v+t)!}{v!(n-v)!(m-v)!}. \end{aligned}$$

### 3.3. Examples

**Example 1.** Let us consider the case when  $K = |\psi\rangle\langle\psi|$  where  $\psi$  is a Schrödinger cat state:

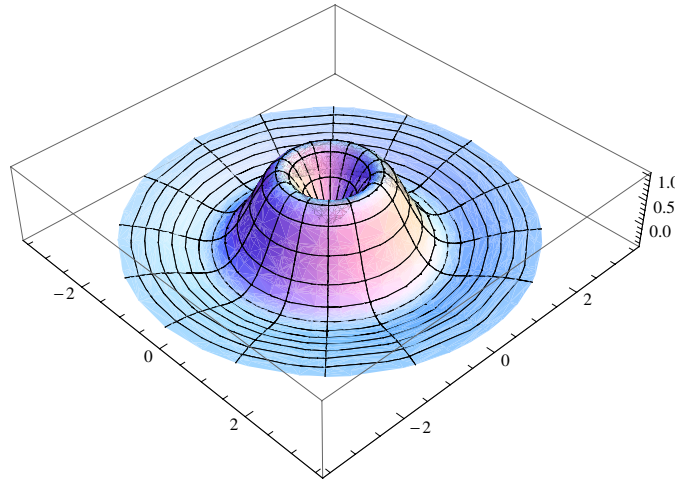
$$\psi = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

From (3.7) we get  $M_{0,0}^{(0)\langle 0|}(\theta, r) = Y(r)$ ,  $M_{0,0}^{(0)\langle 1|}(\theta, r) = -e^{i\theta} Y^{(1)}(r)/\sqrt{2}$ , and  $M_{0,0}^{(1)\langle 1|}(\theta, r) = Y(r) + Y^{(2)}(r)/2$ , so that

$$\begin{aligned} M_{0,0}^{|\psi\rangle\langle\psi|}(\theta, r) &= Y(r) - \frac{1}{\sqrt{2}} Y^{(1)}(r) \cos \theta + \frac{1}{4} Y^{(2)}(r) \\ &= 2(r + \sqrt{2} \cos \theta)[r + (1 - 2r^2) \text{daw}(r)]. \end{aligned}$$

The function  $[0, \pi) \times \mathbb{R} \ni (\theta, r) \mapsto M_{0,0}^{|\psi\rangle\langle\psi|}(\theta, r) \in \mathbb{R}$  is plotted in figure 1 by interpreting  $(\theta, r)$  as the polar coordinates of  $\mathbb{R}^2$ . From the picture we see the effect of the interference term  $(|0\rangle\langle 1| + |1\rangle\langle 0|)/2$  added to the mixed state  $S := (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ : obviously,  $M_{0,0}^S(\theta, r) = Y(r) + Y^{(2)}(r)/4$  is rotation symmetric (does not depend on  $\theta$ ) whereas  $M_{0,0}^{|\psi\rangle\langle\psi|}(\theta, r)$  is  $\theta$ -dependent (see figure 2).

**Example 2.** In this example, we study the Cahill–Glauber  $s$ -parametrized quasiprobability distributions [5]. Instead of using  $s$  as a parameter, we define  $\lambda := (s + 1)/(s - 1)$  and



**Figure 2.** The density  $M_{0,0}^S(\theta, r)$  of the generalized Markov kernel in the case of  $S = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ .

assume that  $\lambda \in [-1, 1)$  (that is,  $s = (\lambda + 1)/(\lambda - 1) \in (-\infty, 0]$ ). Let  $T \in \mathcal{T}(\mathcal{H})$ . Now the  $s$ -distribution is  $p_{K_\lambda}^T$  where the kernel  $K_\lambda := (1 - \lambda) \sum_{n=0}^\infty \lambda^n |n\rangle\langle n|$  [5, equation (6.23)]. Note that  $K_\lambda \in \mathcal{T}^S(\mathcal{H})$  only when  $\lambda \in (-1, 1)$ , and then  $\text{tr } K_\lambda = 1$ . For  $\lambda = 0$  (and  $s = -1$ ),  $K_0 = |0\rangle\langle 0|$ , and  $p_{K_0}^T$  is the Husimi  $Q$ -function. When  $\lambda = -1$  (and  $s = 0$ ) we have  $K_{-1} = 2\Pi$  and  $p_{K_{-1}}^T = p_{2\Pi}^T = 2\pi W^T$ . Note that for  $\lambda \in (-1, 1)$ , one can define an operator measure  $E_{K_\lambda}$  and calculate  $M_{0,0}^{K_\lambda}(\theta, r)$  as before, but when  $\lambda = -1$  the corresponding  $E_{K_{-1}}$  is only a generalized operator measure, that is, a sesquilinear form-valued measure [18]. Hence, we assume that  $\lambda \in (-1, 1)$  and finally take the limit  $\lambda \rightarrow -1$ .

By theorem 1, one sees that

$$\begin{aligned} M_{0,0}^{K_\lambda}(\theta, r) &= (1 - \lambda) \sum_{n=0}^\infty \lambda^n M_{0,0}^{[n]\langle n|}(\theta, r) = (1 - \lambda) \sum_{n=0}^\infty \frac{(-\lambda)^n}{n!} \sum_{k=n}^\infty \frac{(-1)^k}{2^k (2k)!} \frac{(k!)^2}{(k-n)!} H_{2k}(r) \\ &= (1 - \lambda) \sum_{k=0}^\infty \frac{(-1)^k k!}{2^k (2k)!} H_{2k}(r) \underbrace{\sum_{n=0}^k \frac{(-\lambda)^n k!}{n!(k-n)!}}_{=(1-\lambda)^k} = (1 - \lambda) \sum_{k=0}^\infty \frac{(\lambda - 1)^k k!}{2^k (2k)!} H_{2k}(r). \end{aligned}$$

It is easy to calculate the Maclaurin series of  $r \mapsto M_{0,0}^{K_\lambda}(\theta, r)$  (see equation (A.1) and the text above [12, equation (A.6)]): we get

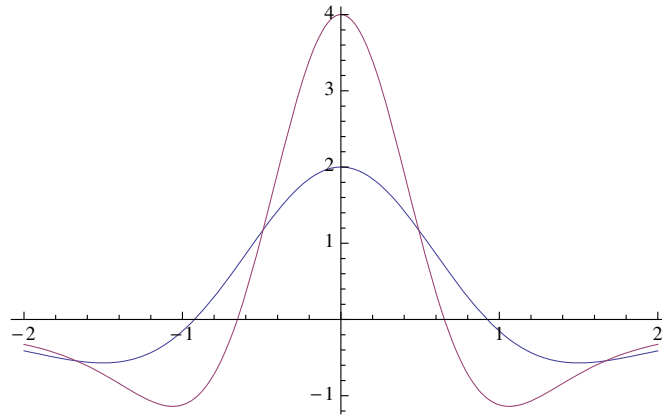
$$M_{0,0}^{K_\lambda}(\theta, r) = 2 \sum_{u=0}^\infty \frac{(-1)^u u! (2r)^{2u}}{(2u)!} \underbrace{\left( \frac{1 - \lambda}{1 + \lambda} \right)^{u+1}}_{=-1/s}.$$

From equation (3.8) one sees that

$$M_{0,0}^{K_\lambda}(\theta, r) = -\frac{1}{s} Y(r/\sqrt{-s}) = -\frac{1}{s} M_{0,0}^{[0]\langle 0|}(\theta, r/\sqrt{-s}).$$

When  $\lambda \rightarrow -1, s \rightarrow 0$ ,

$$M_{0,0}^{K_\lambda}(\theta, 0) = -\frac{1}{s} Y(0) = \frac{2}{-s} \rightarrow \infty$$



**Figure 3.** The density  $r \mapsto M_{0,0}^{K_\lambda}(\theta, r)$  of the generalized Markov kernel (for all  $\theta \in [0, 2\pi)$ ). The higher curve corresponds to the case  $s = -\frac{1}{2}$  and for the lower curve  $s = -1$ .

so that the density of the Markov kernel is not defined in the Wigner function limit, as expected. Note that  $M_{0,0}^{K_\lambda}$  does not define the Dirac  $\delta$ -distribution in the limit (even though it is an increasing peak, see figure 3), since  $\int_{\mathbb{R}} Y(r) dr = 0$  implies that

$$\int_{\mathbb{R}} M_{0,0}^{K_\lambda}(\theta, r) dr = 0$$

for all  $\lambda \in (-1, 1)$ . However, for any  $(q_1, p_1) \in \mathbb{R}^2$  one gets

$$2\pi W^T(q_1, p_1) = p_{K_{-1}}^T(q_1, p_1) = \lim_{\lambda \rightarrow -1^+} \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} M_{q_1, p_1}^{K_\lambda}(\theta, r) p_{\text{ht}}^T(\theta, r) d\theta dr$$

but cannot change the order of taking the limit and integration.

**Example 3.** In this example, we show that the generalized Markov kernel does not exist for all states. Choose  $K_\eta = |\eta\rangle\langle\eta|$  where  $\eta = \chi_{[0,1]} \in L^2(\mathbb{R})$  (the characteristic function of  $[0, 1]$ ) is a unit vector. The first observations are that  $K_\eta \notin \mathcal{T}^S(\mathcal{H})$  and the Wigner function  $W^{K_\eta}$  of  $K_\eta$  is not integrable. Thus, the equation  $(\text{RW}^{K_\eta})(\theta, r) = p_{\text{ht}}^{K_\eta}(\theta, r)$  is not necessarily valid. However, one can calculate  $p_{\text{ht}}^{K_\eta}(\theta, r)$ . For example,  $p_{\text{ht}}^{K_\eta}(0, r) = [\chi_{[0,1]}(r)]^2 = \chi_{[0,1]}(r)$  and we can try to derive  $M_{q_1, p_1}^{K_\eta}(\theta, r)$  as before we did for  $\mathcal{T}^S(\mathcal{H})$ -kernels. But now  $d\chi_{[0,1]}(x)/dx = \delta(x) - \delta(x - 1)$  must be understood as a (tempered) distribution and a formal calculation shows that

$$M_{0,0}^{K_\eta}(0, r) = \pi \left( \text{H} \frac{d\chi_{[0,1]}}{dx} \right)(r) = \frac{1}{r} - \frac{1}{r-1}$$

which is singular at  $r = 0$  and  $r = 1$ . On the other hand,

$$\pi(\text{H}\chi_{[0,1]})(r) = \ln \left| \frac{r}{r-1} \right|$$

and we get the same result:

$$M_{0,0}^{K_\eta}(0, r) = \frac{d}{dr} \ln \left| \frac{r}{r-1} \right| = \frac{1}{r} - \frac{1}{r-1}.$$



It is interesting to note that  $E_{K_\eta}$  is not informationally complete, since the support of

$$(q, p) \mapsto \text{tr} [K_\eta D(q, p)] = \langle \eta | D(q, p) \eta \rangle = e^{-iqp/2} \int_0^1 e^{ipx} \chi_{[0,1]}(x - q) dx$$

is not the whole  $\mathbb{R}^2$  [3, 14]. It is an open question how the informational completeness of a phase-space observable  $E_K$  is connected with the existence of the generalized Markov kernel  $\mathcal{M}_K$ .

**Acknowledgments**

The author thanks Pekka Lahti for valuable discussions and for the carefully reading of the paper.

**Appendix. Radon and Hilbert transforms**

The basic properties of the Radon and Hilbert transforms,  $R$  and  $H$ , respectively, are collected in this appendix. There is plenty of literature on these transforms. For example, references [6, 8] essentially contain the results for  $R$  whereas [4, 17] are good references for  $H$ . The elementary results of this appendix which are easy to prove (e.g., by changing integration variables and by using Fubini’s theorem), are denoted by  $(\star)$ .

Let  $\mathbb{P}^2$  be the set of straight lines in  $\mathbb{R}^2$ . Any line  $\xi(\theta, r) := \{(x, y) \in \mathbb{R}^2 \mid x \cos \theta + y \sin \theta = r\}$  can be parametrized with two real numbers  $\theta \in [0, 2\pi)$  and  $r \in \mathbb{R}$ , and  $\mathbb{P}^2$  becomes a smooth two-dimensional manifold. Since  $\xi(-\theta, -r) = \xi(\theta, r)$ ,  $\mathbb{P}^2$  has a double covering  $\mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{P}^2$  where  $\mathbb{S}^1$  is the unit circle  $\{(\cos \theta, \sin \theta) \in \mathbb{R}^2 \mid \theta \in [0, 2\pi)\}$ . An unambiguous parametrization of  $\mathbb{P}^2$  is then given with the same parameters as above but restricting their domains to be either  $\theta \in [0, \pi)$  and  $r \in \mathbb{R}$ , or  $\theta \in [0, 2\pi)$  and  $r \geq 0$ . From now on we use the double covering space  $\mathbb{S}^1 \times \mathbb{R}$  instead of  $\mathbb{P}^2$  and, for example, identify functions  $\mathbb{P}^2 \rightarrow \mathbb{C}$  with functions  $f : [0, 2\pi) \times \mathbb{R} \rightarrow \mathbb{C}$  with the property  $f(-\theta, -r) = f(\theta, r)$  for all  $\theta \in [0, 2\pi)$  and  $r \in \mathbb{R}$ . Moreover, without any further mention, we equip  $\mathbb{S}^1 \times \mathbb{R}$  with the measure  $(2\pi)^{-1} d\theta dr$ .

Let  $\mathcal{S}(\mathbb{P}^2)$  rapidly decreasing smooth functions  $\mathbb{P}^2 \rightarrow \mathbb{C}$  (obviously in the direction of growing  $r$ ). Let  $\mathcal{S}_H(\mathbb{P}^2)$  consist of functions  $\varphi \in \mathcal{S}(\mathbb{P}^2)$  such that, for all  $k \in \mathbb{Z}_+$ , the mapping  $\theta \mapsto \int_{\mathbb{R}} \varphi(\theta, r) r^k dr$  is a homogeneous polynomial of degree  $k$  with respect to ‘variables’  $\cos \theta$  and  $\sin \theta$ .

The *Radon transformation* is a continuous  $(\star)$  linear mapping  $R : L^1(\mathbb{R}^2) \rightarrow L^1(\mathbb{S}^1 \times \mathbb{R})$  given by

$$(Rf)(\theta, r) := \int_{\mathbb{R}} f(r \cos \theta - t \sin \theta, r \sin \theta + t \cos \theta) dt$$

for  $d\theta dr$ -almost all  $(\theta, r)$ ; here, we consider the above  $L^1$ -spaces as Banach spaces equipped with the standard  $L^1$ -norms. We have the following Schwartz theorem [8, theorem 2.4]:

$$R|_{\mathcal{S}(\mathbb{R}^2)} : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}_H(\mathbb{P}^2) \text{ is bijective.}$$

The operator  $R$  has an adjoint operator, the so-called *backprojection operator*,  $R^*$  defined as

$$(R^*\varphi)(x, y) := \int_0^{2\pi} \varphi(\theta, x \cos \theta + y \sin \theta) \frac{d\theta}{2\pi}$$

where, e.g.,  $\varphi \in L^1(\mathbb{S}^1 \times \mathbb{R})$ . Indeed, for suitable classes of functions  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{C}$  one can easily verify  $(\star)$  that  $\langle Rf | \varphi \rangle_{L^2(\mathbb{S}^1 \times \mathbb{R})} = \langle f | R^*\varphi \rangle_{L^2(\mathbb{R}^2)}$ .

Let  $f, g \in L^1(\mathbb{R}^2)$ , and define the convolution  $f * g \in L^1(\mathbb{R}^2)$  as

$$(f * g)(x, y) := \iint_{\mathbb{R}^2} f(x - t, y - u)g(t, u) dt du.$$

We have the following *convolution theorem* ( $\star$ ):

$$[\mathbf{R}(f * g)](\theta, r) = \int_{\mathbb{R}} (\mathbf{R}f)(\theta, r - s)(\mathbf{R}g)(\theta, s) ds. \tag{A.1}$$

Define the *Hilbert transform*  $\mathbf{H}$  which is a bounded linear operator on  $L^2(\mathbb{R})$  given by

$$(\mathbf{H}\psi)(r) := \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{\psi(x)}{r-x} dx = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{\infty} \frac{\psi(r+y) - \psi(r-y)}{y} dy$$

for almost all  $r \in \mathbb{R}$  and the above limit exists also in the  $L^2$ -norm [4, theorem 8.1.7].<sup>11</sup> The Hilbert transform has the following properties:

(1)  $\mathbf{H}$  is shift invariant ( $\star$ ), that is,

$$(\mathbf{H}\psi_a)(r) = (\mathbf{H}\psi)(r - a), \quad a, r \in \mathbb{R},$$

where  $\psi \in L^2(\mathbb{R})$  and  $\psi_a(x) := \psi(x - a)$  for all  $x \in \mathbb{R}$ .

(2)  $\mathbf{H}$  commutes with derivation [4, proposition 8.3.7], that is,

$$\mathbf{H}\left(\frac{d\psi}{dx}\right) = \frac{d\mathbf{H}\psi}{dr},$$

where  $\psi \in L^2(\mathbb{R})$  is absolutely continuous and  $d\psi/dx \in L^2(\mathbb{R})$  (that is, when  $\psi$  belong to the domain of the momentum operator  $P$ ).

For any  $\varphi \in \mathcal{S}_H(\mathbb{P}^2)$  define  $\varphi_\theta(x) := \varphi(\theta, x)$  and

$$(\Lambda\varphi)(\theta, r) := \pi \mathbf{H}\left(\frac{d\varphi_\theta}{dx}\right)(r) = \lim_{\epsilon \rightarrow 0^+} \int_{\substack{x \in \mathbb{R} \\ |r-x| > \epsilon}} \frac{1}{r-x} \frac{\partial \varphi(\theta, x)}{\partial x} dx = \pi \frac{d}{dr} (\mathbf{H}\varphi_\theta)(r). \tag{A.2}$$

We have the *Radon inversion theorem* [8, theorem 3.6]: for all  $f \in \mathcal{S}(\mathbb{R}^2)$ ,

$$f = \frac{1}{2\pi} \mathbf{R}^* \Lambda \mathbf{R} f, \tag{A.3}$$

which implies the following *Plancherel formula*:

$$\begin{aligned} \iint_{\mathbb{R}^2} \overline{f(x, y)} g(x, y) dx dy &= \langle f | g \rangle_{L^2(\mathbb{R}^2)} = \frac{1}{2\pi} \langle f | \mathbf{R}^* \Lambda \mathbf{R} g \rangle_{L^2(\mathbb{R}^2)} = \frac{1}{2\pi} \langle \mathbf{R} f | \Lambda \mathbf{R} g \rangle_{L^2(\mathbb{S}^1 \times \mathbb{R})} \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_0^{2\pi} \overline{(\mathbf{R}f)(\theta, r)} (\Lambda \mathbf{R}g)(\theta, r) d\theta dr \end{aligned} \tag{A.4}$$

for all  $f, g \in \mathcal{S}(\mathbb{R}^2)$ .

## References

- [1] Abramowitz M and Stegun I A (ed) 1972 *Handbook of Mathematical Functions (National Bureau of Standards, Applied Mathematics Series vol 55)* (10th printing with corrections) (Washington, DC: National Bureau of Standards)
- [2] Albin P, De Vito E and Toigo A 2009 Quantum homodyne tomography as an informationally complete positive-operator-valued measure *J. Phys. A: Math. Theor.* **42** 295302

<sup>11</sup> Note that some authors multiply  $\mathbf{H}$  by the imaginary unit  $i$  in the definition of the Hilbert transform to get a self-adjoint operator.

- [3] Ali S T and Prugovečki E 1977 Classical and quantum statistical mechanics in a common Liouville space *Physica* **89A** 501–21
- [4] Butzer P L and Nessel R J 1971 *Fourier Analysis and Approximation* vol 1 (Basel: Birkhäuser)
- [5] Cahill K E and Glauber R J 1969 Ordered expansions in boson amplitude operators *Phys. Rev.* **177** 1857–81
- [6] Epstein C L 2008 *Introduction to the Mathematics of Medical Imaging* 2nd edn (Philadelphia: SIAM)
- [7] Feldheim E 1938 Quelques nouvelles relations pour les polynomes d’Hermite *J. Lond. Math. Soc.* **13** 22–9
- [8] Helgason S 1999 *The Radon Transform* 2nd edn *Progress in Mathematics* (Boston: Birkhäuser)
- [9] Holevo A S 1985 Statistical definition of observable and the structure of statistical models *Rep. Math. Phys.* **22** 385–407
- [10] Ibrat A, Man’ko V I, Marmo G, Simoni A and Ventriglia F 2009 An introduction to the tomographic picture of quantum mechanics *Phys. Scr.* **79** 065013
- [11] Kiukas J and Lahti P 2008 A note on the measurement of phase-space observables with an eight-port homodyne detector *J. Mod. Opt.* **55** 1891–8
- [12] Kiukas J, Lahti P and Pellonpää J-P 2008 A proof for the informational completeness of the rotated quadratures *J. Phys. A: Math. Theor.* **41** 175260
- [13] Kiukas J, Lahti P and Schultz J 2009 Position and momentum tomography *Phys. Rev. A* **79** 052119
- [14] Kiukas J and Werner R 2008 private communication
- [15] Lahti P and Pellonpää J-P 2009 Continuous variable tomographic measurements *Phys. Lett. A* **373** 3435–8
- [16] Leonhardt U 1997 *Measuring the Quantum State of Light* (Cambridge: Cambridge University Press)
- [17] Pandey J N 1996 *The Hilbert Transform of Schwartz Distributions and Applications* (New York: Wiley)
- [18] Pellonpää J-P 2001 Phase observables, phase operators and operator orderings *J. Phys. A: Math. Gen.* **34** 7901–16
- [19] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics: I. Functional Analysis* (New York: Academic)
- [20] Riordan J 1968 *Combinatorial Identities* (New York: Wiley)
- [21] Spanier J and Oldham K B 1987 *An Atlas of Functions* (New York: Hemisphere)